Re-Entry Dynamics of a Trimmed Body with Constant Spin

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An analytical definition of the oscillatory motion of a body spinning at a high and constant rate and having a trim angle of attack which can vary with altitude is obtained using the method of stationary phase. The results indicate that for trim angles of constant sign the first maximum angle of attack is directly proportional to the trim magnitude, spin rate, and moment of inertia and inversely proportional to velocity and flight path angle. The effect of both initial body misalignment and tumble rate are included and the results show that the bounds on the maximums of angle of attack are obtained from the sum of the individual absolute values of the trim and the no-trim contributions. The condition of misalignment with no tumble rate is investigated in detail and simplified relations are developed which define the trimmed body dynamics. The analytical results are compared with numerical solutions, and in general good agreement is found over most of the trajectory using the simplified relations.

Nomenclature

A,B = constants of integration, [Eqs. (8) and (9)]
b_n = constant [see Eq. (5)]
$C_{m\alpha}$ = aerodynamic moment coefficient slope
d = reference length
$g(\zeta)$ = [see Eq. (61)]
I = $I_x = I_y$
I_x,I_y,I_z = moment of inertia about pitch, yaw and roll axis
K_2 = $\rho_0 Sd(-C_{m\alpha})/2I\beta^2 \sin^2 \gamma$ = static stability parameter
K_3 = $p_0 I_z/\beta IV \sin\gamma$ = spin parameter
K_n = [see Eq. (7)]
p_0 = spin rate
q_E,r_E = components of initial angular velocity along the pitch and yaw axis, respectively
S = reference area
t = time
V = vehicle velocity = constant
y = altitude
$Z(\pm\zeta)$ = [Eq. (15)], $Z(\zeta/\zeta_r)$ = [see Eq. (23)], $Z(\zeta/\zeta_A)$ = [see Eq. (40)]
α = complex angle of attack = $\theta e^{-i\psi}$
α_E = complex initial angle of attack
$\alpha_F(\zeta)$ = resonant contribution to angle of attack [see Eq. (10)]
$\alpha_H(\zeta)$ = homogeneous contribution to angle of attack [see Eq. (4)]
α_{T_n} = constant, [see Eq. (5)]
$\alpha_T(\zeta)$ = trim angle of attack [see Eq. (5)]
β = scale height = constant in density altitude relation
γ = flight-path angle from horizontal = constant
ζ = $K_2^{1/2}e^{-\beta y/2}$, ζ_A = value for termination of stationary region, [Eq. (35)], ζ_M = value for first maximum angle of attack, [Eq. (33)]
ζ_r = $K_3(1-\lambda)/2(1+\lambda)$
$\zeta^*_{(\ )}$ = $\zeta_{(\ )}/\zeta_r$
η = [see Eq. (36)]
θ_B,θ_F = [see Eqs. (62) and (65)]
ϕ,θ,ψ = Eulerian angles (see Fig. 1)
λ = $(I_z/I) - 1$
$\mu(\zeta)$ = instantaneous frequency in cycles per unit ζ
$\mu(t)$ = instantaneous frequency in cycles per unit time
ξ = [see Eq. (39)]
ρ = $\rho_0 e^{-\beta y}$ = air density, ρ_0 = sea level value
ω_E = $q_E + ir_E$ = initial tumble rate
$1,i$ = unit vectors (see Fig. 1)

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Superscripts

$(\dot{\ })$ = d/dt
$(\ )'$ = $d/d\zeta$

Subscripts

F,H = forced resonant and homogeneous contributions, Eq. (10)
$\mathcal{I},\mathcal{R}$ = imaginary and real components, respectively

Introduction

During atmospheric re-entry a spinning body with a trim angle of attack, such as that which might be induced by aerodynamic configuration asymmetries will encounter a trim amplification which is strongly dependent on the magnitude of the ratio of the spin frequency to the body natural or critical frequency.[1,2] In the vicinity of equal spin and critical frequencies, the body angle of attack can become very large and result in a significant perturbation to the subsequent trajectory history. Although digital machine simulations are generally used to predict angle-of-attack divergence in the region of transient amplification, a simplified analytical description of the major contributing parameters is of interest.

A number of reports have treated various aspects of this problem. Nicolaides[1] and Nelson[2] considered constant altitude and constant velocity solutions showing the amplification of trim and the body oscillatory characteristics as a function of the ratio of spin to critical frequency. Kanno[3] considered a time varying environment for slender bodies perfectly aligned with the velocity vector at re-entry and having constant trim. He provided a series solution which was evaluated numerically.

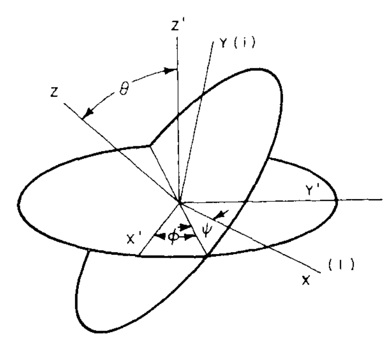

Fig. 1 Coordinate system.

In the present paper approximate analytical relations are derived which define the oscillatory motion of a spinning reentry body with trim. The solutions allow evaluations of initial body misalignment and tumble rate and include a trim which can vary with altitude. The approach used is to add a trim contribution to Leon's[4] equation for the oscillatory behavior of a symmetric body with no trim. The solutions are obtained for constant velocity trajectories using the method of stationary phase which provides asymptotic approximations of the integrals defining resonant excitation. The equations are evaluated in pieces according to the trajectory region of interest and simplifying assumptions are provided to allow gross indications of body behavior.

General Equation

By adding a trim contribution to Leon's results,[4] the equation of rotational motion becomes

$$\ddot{\alpha} + i(1 - \lambda)p_0\dot{\alpha} + \{\lambda p_0^2 + [\rho V^2 Sd(-C_{m\alpha})/2I]\}\alpha = \rho V^2 Sd(-C_{m\alpha})\alpha_T(t)/2I \quad (1)$$

where the Eulerian angles describing the angular orientation of the vehicle are shown in Fig. 1. The derivation assumes negligible aerodynamic damping, constant velocity, flight-path angle and spin rate, zero products of inertia and an exponential density variation. The angle of attack α is assumed to be small throughout the trajectory, and the trim angle of attack α_T may be variable within certain limitations specified in the discussion of the high-altitude solution and the stationary region. The general integral equation is obtained by using the change of variable $\zeta = K_2^{1/2}e^{-\beta y/2}$, and Eq. (1) becomes

$$\zeta^2\alpha'' + \zeta[1 + 4\zeta_r]\alpha' + [4K_3^2\lambda/(1+\lambda)^2 + 4\zeta^2] = 4\zeta^2\alpha_T(\zeta) \quad (2)$$

which has the solution[5]

$$\alpha(\zeta) = -i\frac{2\pi}{\sinh \pi K_3}e^{-i2\zeta_r \ln\zeta}\left\{J_{iK_3}(2\zeta)\int_0^\zeta \alpha_T(\zeta)\zeta^{1+i2\zeta_r} \times J_{-iK_3}(2\zeta)d\zeta - J_{-iK_3}(2\zeta)\int_0^\zeta \alpha_T(\zeta)\zeta^{1+i2\zeta_r}J_{iK_3}(2\zeta)d\zeta\right\} + \alpha_H(\zeta) \quad (3)$$

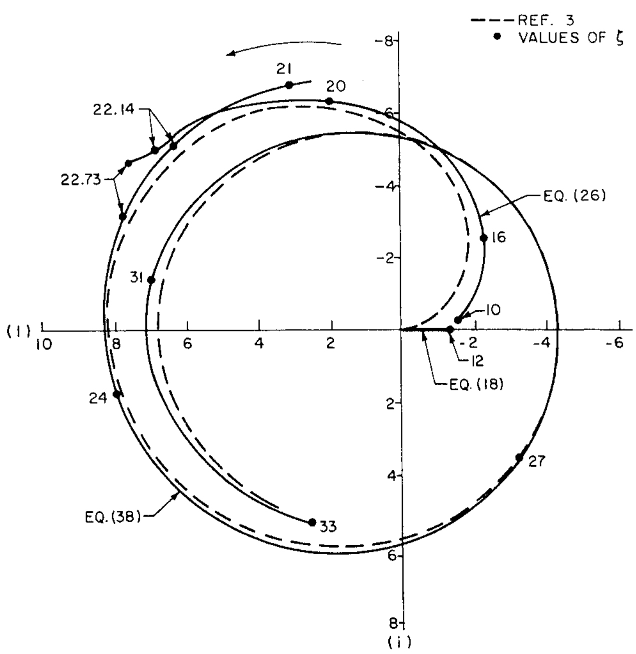

Fig. 2 Angle-of-attack history.

where the homogeneous solution is

$$\alpha_H(\zeta) = e^{-i2\zeta_r \ln\zeta}\{AJ_{iK_3}(2\zeta) + BJ_{-iK_3}(2\zeta)\} \quad (4)$$

and A and B are constants defined by initial re-entry conditions.

The constants are evaluated by considering the high-altitude ($\zeta \to 0$) solution for a trim history of the form[5]

$$\alpha_T(\zeta) = \sum_{n=0}^{N}\alpha_{T_n}\zeta^{b_n},\ b_n > -2 \quad (5)$$

where α_T can be either real or complex; however, in the present analysis only real values are used.

The angle of attack at high altitudes is[5]

$$\alpha(\zeta) = \sum_{n=0}^{N}K_n\alpha_{T_n}\zeta^{b_n+2} + \frac{A\zeta^{i[2K_3\lambda/(1+\lambda)]}}{\Gamma(1+iK_3)} + \frac{B\zeta^{-i[2K_3/(1+\lambda)]}}{\Gamma(1-iK_3)} \quad (6)$$

where

$$K_n = [(b_n+2)^2/4 + \lambda K_3^2/(1+\lambda)^2 + i\zeta_r(b_n+2)]^{-1} \quad (7)$$

The first term of Eq. (6) is the high-altitude solution of the nonhomogeneous contribution, and the second two terms are the homogeneous contribution $\alpha_H(\zeta \to 0)$.

The complete solution for $b_n > -2$ is the homogeneous solution, and the constants are[4]

$$A = -i[\omega_E/p_0(1+\lambda)]K_2^{-iK_3\lambda/(1+\lambda)}\Gamma(1+iK_3) \quad (8)$$

and

$$B = [\alpha_E + i\omega_E/p_0(1+\lambda)]K_2^{iK_3/(1+\lambda)}\Gamma(1-iK_3) \quad (9)$$

The high-altitude solution shows that for a trim history defined by Eq. (5) the body trim does not effect the homogeneous contribution, and that the α history may be separated and evaluated independently according to

$$\alpha(\zeta) = \alpha_F(\zeta) + \alpha_H(\zeta) \quad (10)$$

where $\alpha_F(\zeta)$ is the resonant contribution as defined by the terms other than $\alpha_H(\zeta)$, in Eq. (3) and $\alpha_H(\zeta)$ is given by Eq. (4).

The complete general integral solution is therefore given by Eq. (3) with Eq. (4) defining the homogeneous contribution, Eq. (5) defining the trim history used here with the limiting constraint on trim ($b_n > -2$), and the constants of integration defined by Eqs. (8) and (9).

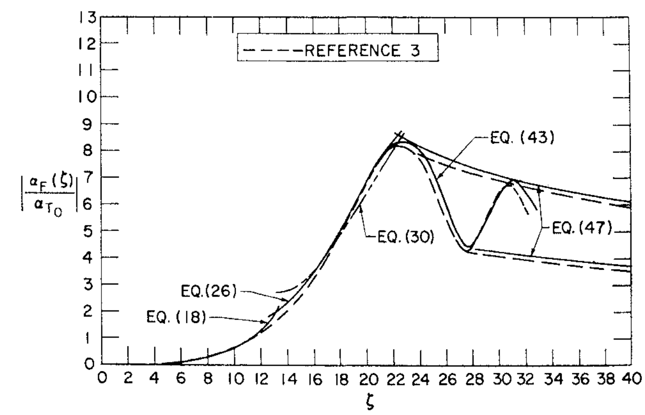

Fig. 3 Absolute value of angle of attack.

Low-Altitude Solution of $\alpha_F(\zeta)$

At low altitudes, ζ is large and the asymptotic value of the Bessel functions for large arguments is[4]

$$J_{\pm iK_3}(2\zeta) \approx (\pi\zeta)^{-1/2}\cos(2\zeta \mp i\pi K_3/2 - \pi/4) \quad (11)$$

The approximation

$$\sinh(\pi K_3/2) \approx \cosh(\pi K_3/2) \quad (12)$$

is valid for large spin rates. The Bessel function can be approximated by

$$J_{\pm iK_3}(2\zeta) \approx (\pi\zeta)^{-1/2} e^{\pm i(2\zeta - \pi/4)} \sinh(\pi K_3/2) \quad (13)$$

Substituting Eq. (13) into the resonant contribution of Eq. (3) results in

$$\alpha_F(\zeta) = -i\zeta^{-1/2}e^{-i2\zeta_r Z(-\zeta)} \left\{ \int_0^\zeta \alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(-\zeta)}d\zeta - e^{-i4\zeta}\int_0^\zeta \alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(+\zeta)}d\zeta \right\} \quad (14)$$

where

$$Z(\pm\zeta) = \ln\zeta \pm \zeta^* \quad (15)$$

Approximate solutions to Eq. (14) are available using the method of stationary phase, which applies if ζ_r is large.[6] The solutions are divided according to regions which do and do not contain a stationary point. For spin rates which are positive only the first integral of Eq. (14) contains a stationary point during re-entry. The stationary point is obtained by setting $dZ(-\zeta)/d\zeta = 0$, resulting in

$$\zeta = \zeta_r \quad (16)$$

The discussion of the low-altitude analysis is therefore subdivided according to the three successive regions where the solutions to the integrals apply: prestationary, stationary and poststationary.

Prestationary Region

The solution to the integrals of Eq. (14) in an interval containing no stationary point is[6]

$$\int_{\zeta_1}^{\zeta_2} \alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(\pm\zeta)}d\zeta = \frac{-i\alpha_T(\zeta_2)\zeta_2^{1/2}e^{i2\zeta_r Z(\pm\zeta_2)}}{2\zeta_r Z'(\pm\zeta_2)} + \frac{i\alpha_T(\zeta_1)\zeta_1^{1/2}e^{i2\zeta_r Z(\pm\zeta_1)}}{2\zeta_r Z'(\pm\zeta_1)} + 0\left(\frac{1}{4\zeta_r^2}\right) \quad (17)$$

Using Eq. (17) in Eq. (14) the angle of attack at any altitude prior to the stationary region can be obtained from

$$\alpha_F(\zeta) = -\alpha_T(\zeta)/(\zeta^{*-2} - 1) \quad (18)$$

where the remainder has been dropped, and the lower limit in the integration is $\zeta_1 = 0$.

The results using Eq. (18) are compared with Kanno's numerical data[3] in Figs. 2 and 3 for a body with $\zeta_r = 16$, a constant trim and no re-entry misalignment, $\alpha_H(\zeta) = 0$. The results of the comparison show good agreement in angular displacement, except near the stationary region. The stationary region is pictorially described in Fig. 4 using the real and imaginary components of the exponential of the first integral of Eq. (14). The poor agreement in this region is expected since the behavior of both real and imaginary components of the function under the integral are no longer cyclic.

The small change in angular orientation of $\alpha_F(\zeta)$ given by Ref. 3 is not predicted by the asymptotic approximation, as shown in Fig. 2; however, the prediction may be improved, though not without an increase in complexity, as described in the discussion of the stationary region.

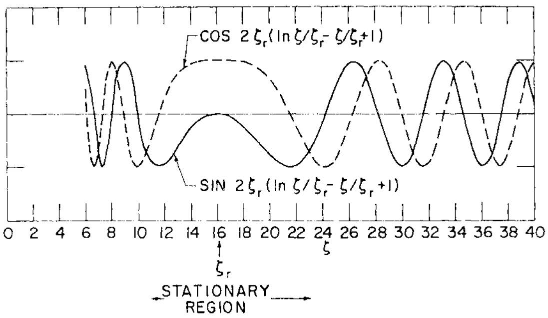

Fig. 4 Behavior of the normalized exponential containing the stationary point.

Stationary Region $-\alpha_F(\zeta)$

The stationary region, Fig. 4, is described as the region where the cyclic nature of exponent $i(\ln\zeta - \zeta^*)$ is absent. The value of the integral containing the stationary point at $\zeta = \zeta_r$ can be approximated for any region which contains the stationary point by integrating over an interval which has the stationary point on either the left or the right end of that interval.[6] The result for large ζ_r is a constant of the form

$$\int_{\zeta_1}^{\zeta_2} \alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(-\zeta)}d\zeta = \pi^{1/2}\{-4\zeta_r Z''(-\zeta_r)\}^{-1/2} \times \{\alpha_T(\zeta_r)\zeta_r^{1/2}e^{i[2\zeta_r Z(-\zeta_r) - \pi/4]}\} + 0\left(\frac{1}{2\zeta_r}\right) \quad (19)$$

for $\alpha_T(\zeta_r) \neq 0$.

The solution, Eq. (19), is identical if either ζ_1 or ζ_2 is set at ζ_r. The angle of attack at the stationary point is obtained by using Eq. (19) for the first integral of Eq. (14) and Eq. (17) for the second integral, where $\zeta_1 = 0$ and $\zeta_2 = \zeta_r$. The result at the stationary point $\zeta = \zeta_r$ is

$$\alpha_F(\zeta_r)/\alpha_T(\zeta_r) = \tfrac{1}{4}[1 - (2\pi\zeta_r)^{1/2} - i(2\pi\zeta_r)^{1/2}] \quad (20)$$

and the absolute value is

$$|\alpha_F(\zeta_r)/\alpha_T(\zeta_r)| = \tfrac{1}{4}[1 + 4\pi\zeta_r - (8\pi\zeta_r)^{1/2}]^{1/2} \quad (21)$$

The angle of attack in the stationary region is obtained by considering two integration intervals for the first or stationary integral of Eq. (14): the first integration interval between $\zeta_1 = 0$ and $\zeta_2 = \zeta_r$ and the second interval between $\zeta_1 = \zeta_r$ and $\zeta_2 = \zeta$. For the first integration interval Eq. (19) is used directly and Eq. (17) is used for the second integral between the limits $\zeta_1 = 0$ and $\zeta_2 = \zeta$. The result is

$$\alpha_F(\zeta) = -i\zeta^{-1/2}e^{-i2\zeta_r Z(\zeta^*)} \left\{ \left(\frac{\pi}{8}\right)^{1/2} \zeta_r\alpha_T(\zeta_r)(1 - i) + \int_{\zeta_r}^{\zeta} \alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(\zeta^*)}d\zeta + \frac{i[\alpha_T(\zeta)\zeta^{3/2}e^{i2\zeta_r Z(\zeta^*)}]}{2(\zeta_r + \zeta)} \right\} \quad (22)$$

where

$$Z(\zeta^*) = \ln\zeta^* - \zeta^* + 1 \quad (23)$$

For detailed investigation of the region close to $\zeta = \zeta_r$ the integral in Eq. (22) can be evaluated using the series expansions

$$\ln x = \ln x_0 + [(x/x_0) - 1] - \tfrac{1}{2}[(x/x_0) - 1]^2 + \tfrac{1}{3}[(x/x_0) - 1]^3 \ldots \quad (24)$$

for values of x between zero and two, and

$$e^x = 1 + x + (x^2/2!) + (x^3/3!) \quad (25)$$

Using the first three terms of Eqs. (24) and (25) in the integral, Eq. (22) becomes

$$\alpha_F(\zeta) = -i\zeta^{-1/2}e^{i2\zeta_r Z(\zeta^*)}\left\{\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r)(1 - i) + \int_{\zeta_r}^{\zeta}\alpha_T(\zeta)\zeta^{1/2}[1 - i\zeta_r(\zeta^* - 1)^2 - \tfrac{1}{2}\zeta_r^2(\zeta^* - 1)^4]d\zeta + \frac{i[\alpha_T(\zeta)\zeta^{3/2}\epsilon^{i2\zeta_r Z(\zeta^*)}]}{2(\zeta_r + \zeta)}\right\} \quad (26)$$

and can be integrated directly for any trim history given by Eq. (5).

An increase in the number of terms used in the series expansions, Eq. (24) and (25), would improve the accuracy of the results shown in Figs. 2 and 3. The comparisons in Figs. 2 and 3 are based on the same body described in the discussion of the prestationary region, and the results show good agreement between the stationary point and near the point of maximum angle of attack, suggesting that a near linear growth of α_F is a reasonable approximation. For higher altitude regions, $\zeta < \zeta_r$, the solution, Eq. (26), begins to diverge from the numerical results of Kanno; and if a detailed investigation of this region is required, additional terms of the series Eqs. (24) and (25) must be used to evaluate Eq. (22).

For lower altitudes $\zeta > \zeta_r$ near the end of the stationary region, or in the example case, near maximum angle of attack, a similar comment is valid.

Equation (26) may be simplified by first considering the bracketed terms in Eq. (22). The integral term is much greater than the third term as shown by rewriting the integral term as

$$\int_{\zeta_r}^{\zeta}\alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(\zeta^*)}d\zeta = \int_{\zeta_r}^{\zeta_r+\Delta}\alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(\zeta^*)}d\zeta + \int_{\zeta_r+\Delta}^{\zeta}\alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(\zeta^*)}d\zeta \quad (27)$$

which can be integrated using Eq. (19) and Eq. (17), respectively. The integral is therefore equal to a constant added to $i[\alpha_T(\zeta)]\,[\zeta^{3/2}e^{i2\zeta_r Z(\zeta^*)}]/2(\zeta_r - \zeta)$. Comparing this variable term to the third term of Eq. (26) shows that for large ζ_r and in a region where ζ is near ζ_r,

$$1/(\zeta_r + \zeta) \ll 1/(\zeta_r - \zeta) \quad (28)$$

for both real and imaginary components, independent of the value of Δ and $\alpha_T(\zeta)$. The third term in Eq. (22) can therefore be neglected, and using two terms of Eq. (24) in Eq. (22) the α history in the stationary region becomes

$$\alpha_F(\zeta) \approx -i\zeta^{-1/2}e^{-i2\zeta_r Z(\zeta^*)}\left\{\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r)(1 - i) + \int_{\zeta_r}^{\zeta}\alpha_T(\zeta)\zeta^{1/2}d\zeta\right\} \quad (29)$$

and the absolute value is

$$|\alpha_F(\zeta)| \approx \zeta^{-1/2}\left\{\left[\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r) + \int_{\zeta_r}^{\zeta}\alpha_T(\zeta)\zeta^{1/2}d\zeta\right]^2 + \left[\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r)\right]^2\right\}^{1/2} \quad (30)$$

The results using Eq. (30) are compared with the results of Ref. 3 in Fig. 3, and comments similar to those relative to the comparisons of Eq. (26) apply here as well. Equation (26) or Eq. (29) can be used to approximate the α_F history to the point where the exponential in the stationary integral becomes cyclic (see Fig. 4) once again, or in the comparison case near the point of maximum angle of attack.

Stationary Region—Altitude of Maximum $\alpha_F(\zeta)$

The parameter defining the altitude of maximum angle of attack within the stationary region after ζ_r is obtained from Eq. (22), with the nonstationary term omitted according to Eq. (28). Setting the derivative of the square of the absolute value to zero results in

$$0 = -\zeta^{-1}\left\{\left[\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r) + \int_{\zeta_r}^{\zeta}C(\zeta)d\zeta\right]^2 + \left[-\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r) + \int_{\zeta_r}^{\zeta}S(\zeta)d\zeta\right]^2\right\} + 2\left\{\left[\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r) + \int_{\zeta_r}^{\zeta}C(\zeta)d\zeta\right]C(\zeta) + \left[-\left(\frac{\pi}{8}\right)^{1/2}\zeta_r\alpha_T(\zeta_r) + \int_{\zeta_r}^{\zeta}S(\zeta)d\zeta\right]S(\zeta)\right\} \quad (31)$$

where

$$C(\zeta) = \alpha_T(\zeta)\zeta^{1/2}\cos[2\zeta_r Z(\zeta^*)]$$
$$S(\zeta) = \alpha_T(\zeta)\zeta^{1/2}\sin[2\zeta_r Z(\zeta^*)] \quad (32)$$

Approximating the value of the integrals using Eq. (19), the relation, Eq. (31), becomes

$$\sin\left[2\zeta_r Z(\zeta^*) + \frac{3\pi}{4}\right] = \left(\frac{\pi}{4}\right)^{1/2}\frac{\alpha_T(\zeta_r)}{\alpha_T(\zeta)}\frac{\zeta_r}{\zeta^{3/2}} \quad (33)$$

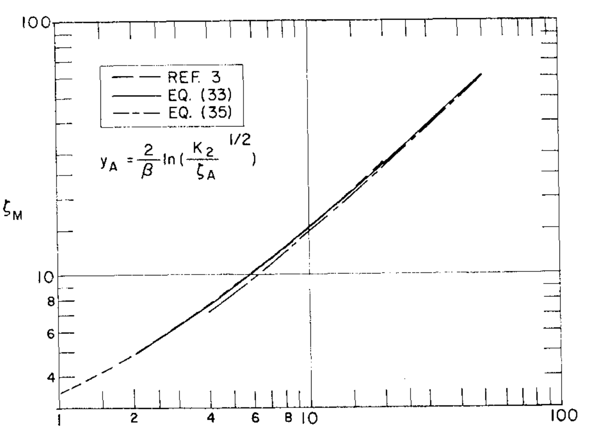

Fig. 5 Altitude of maximum amplification of constant trim.

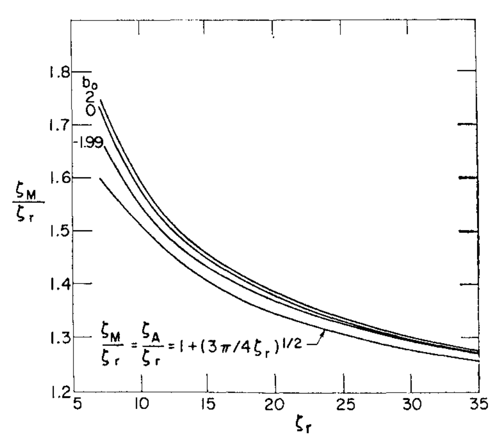

Fig. 6 Effect of variable trim on altitude of first maximum angle of attack.

which is satisfied at the altitude where $\alpha_F(\zeta)$ first reaches a maximum and is designated by the parameter ζ_M.

The altitude of maximum amplification is obtained by dividing Eq. (22) by $\alpha_T(\zeta)$ and following the steps shown in Eqs. (31) and (32). The resulting equation shows that the altitude of maximum amplification within the stationary region is obtained by adding $[\pi^{1/2}\alpha_T'(\zeta)\alpha_T(\zeta_r)\zeta_r]/[\alpha_T^2(\zeta)\zeta^{1/2}]$ to the right-hand side of Eq. (33). For constant trim $\alpha_T'(\zeta) = 0$, and the altitude of maximum amplification coincides with ζ_M.

The comparison of numerical solutions of Eq. (33) with Kanno's results[3] in Fig. 5 indicates good agreement. Figure 6 shows the variation in ζ_M for various values of trim history in terms of b_0 and spin rate ζ_r, for a single term in Eq. (5). The results indicate that b_0 has a small effect on the value of ζ_M over the range of ζ_r and b_0 shown, and that for all b_0, $\zeta_M{}^*$ approaches unity for large ζ_r.

Although a simplified expression defining ζ_M is not known, certain bounding values can be obtained from Eq. (33). For a trim history which does not change sign in the stationary region the right side of Eq. (33) is positive. The first time the equality is satisfied after $\zeta = \zeta_r$ is therefore

$$2\zeta_r Z(\zeta^*) + 3\pi/4 \geq 0 \tag{34}$$

since the argument of the sine is an angle decreasing in magnitude from the value $3\pi/4$ at $\zeta = \zeta_r$. For $\zeta > \zeta_r$ the right side of Eq. (33) goes to zero, approximately according to $\zeta_r{}^{-1/2}$, indicating that ζ_M is near the zero of Eq. (34) for large values of ζ_r. The zero of Eq. (34) is designated as occurring at ζ_A. Using two terms of Eq. (24) in Eq. (34) the altitude at which the zero occurs is approximately

$$\zeta_A \approx \zeta_r + (3\pi\zeta_r/4)^{1/2} \tag{35}$$

and $\zeta_M \approx \zeta_A$ for trim histories which do not change sign in the stationary region. Results using Eq. (35) are also shown in the comparisons of Fig. 5.

For trim histories which change sign in the region between $\zeta_r < \zeta < \zeta_A$, the altitude where the first maximum angle of attack occurs is investigated parametrically for a trim history of the form $\alpha_T(\zeta)/\alpha_T(\zeta_r) = [1 - (\zeta/\zeta_{\alpha_T=0})^{b_1}]/[1 - (\zeta_r/\zeta_{\alpha_T=0})^{b_1}]$ where $\zeta_{\alpha_T=0}$ is the parameter defining the altitude at which the trim goes through zero. The results are presented in terms of η_M and $\eta_{\alpha_T=0}$ in Fig. 7 where

$$\eta(\ ) = [(\zeta)/\zeta_r - 1]/[(\zeta_A/\zeta_r) - 1] \tag{36}$$

is the fraction of the span between ζ_r and ζ_A where the subscript, either $\alpha_T(\zeta) = 0$ or the first maximum angle of attack (subscript M), occurs. The results indicate that for the trim histories evaluated, the altitude of the first maximum angle of attack after ζ_r is primarily dependent on the altitude where the trim goes through zero, and specificially ζ_M preceeds and is close to $\zeta_{\alpha_T=0}$.

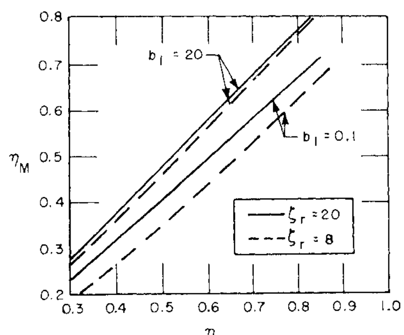

Fig. 7 Effect of zero trim in the stationary region on altitude of first maximum $|\alpha|$.

Stationary Region—Maximum $\alpha_F(\zeta)$

The maximum value of $\alpha_F(\zeta)$ in the stationary region can be obtained by using the results of Eq. (33) in Eq. (30). Maximum $\alpha_F(\zeta)$ as a function of ζ_r is compared with the data of Ref. 3 in Fig. 8, for α_T = constant, and shows good agreement.

For trim histories which do not change sign in the stationary region the effect of vehicle parameters on maximum $\alpha_F(\zeta)$ can be obtained by considering the limiting values of Eqs. (14) and (35) for high ζ_r. As $\zeta_r \to \infty$, the integral with no stationary point can be neglected and the resonant contribution of Eq. (14) can be evaluated using Eq. (19). The amplitude of the first maximum angle of attack, after ζ_r, approaches infinity according to

$$\lim_{\zeta_r \to \infty} |\alpha_F(\zeta_A)| = [\pi\zeta_r\alpha_T(\zeta_r)]^{1/2}$$

since

$$\lim_{\zeta_r \to \infty} (\zeta_A{}^*) = 1$$

Therefore, for high spin rates $\zeta_A \to \zeta_r$ and the ratio of the maximum angle of attack between two bodies which differ slightly in trim history, spin rate, inertias, velocity and flight path angle is approximated from Eq. (30) as

$$\frac{\alpha_F(\zeta_A)}{\alpha_{F_1}(\zeta_{A_1})} \approx \frac{\alpha_T(\zeta_r)}{\alpha_{T_1}(\zeta_{r_1})}\,\frac{p_0(I_z/I)}{p_{0_1}(I_z/I)_1}\,\frac{V_1 \sin\gamma_1}{V \sin\gamma}$$

This ratio along with Eq. (35) shows that for large ζ_r the maximum angle of attack is primarily a function of the magnitude of trim at ζ_r and the altitude at which the maximum angle of attack occurs is relatively independent of the trim history.

For trim histories which change sign in the stationary region, $\zeta_M \neq \zeta_A$ and ζ_M must be obtained from Eq. (33). However, the angle-of-attack history in the stationary region can still be defined by Eq. (26) to the altitude, ζ_A, defined by Eq. (35), even though ζ_M does not coincide with ζ_A. The limitations in the use of Eq. (35) as the upper integration limit are the limitations associated with the number of terms used in Eqs. (24) and (25) to define the integral in Eq. (22). If a large number of terms of Eqs. (24) and (25) were used in Eq. (22), the integration could be extended into the prestationary region limited to altitudes consistent with the assumption, Eq. (11), and from the stationary point to values of ζ up to $2\zeta_r$; however, this approach would, in effect, lead to a numerical approximation. In the present analysis, simplified analytical relations are the goal, and as a result the value of ζ_A defined by Eq. (35) is taken as the termination point of the stationary region, independent of ζ_M, in order to establish an initiation point for the poststationary region solution.

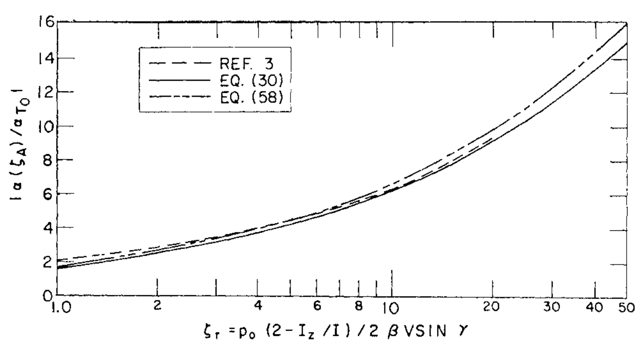

Fig. 8 Maximum amplified angle of attack.

Poststationary Region

Following the stationary region,

$$\alpha_F(\zeta) = -i\zeta^{-1/2}e^{-i2\zeta_r Z(-\zeta)} \left\{\int_0^{\zeta_A} f(-\zeta)d\zeta + \int_{\zeta_A}^{\zeta} f(-\zeta)d\zeta - e^{-i4\zeta}\left[\int_0^{\zeta_A} f(+\zeta)d\zeta + \int_{\zeta_A}^{\zeta} f(+\zeta)d\zeta\right]\right\} \quad (37)$$

where

$$f(\pm\zeta) = \alpha_T(\zeta)\zeta^{1/2}e^{i2\zeta_r Z(\pm\zeta)}$$

The first and third integrals are evaluated as shown in Eq. (26), with ζ_A defined by Eq. (35) and the remaining integrals are evaluated using Eq. (17). The result is

$$\alpha_F(\zeta) = \xi + (\zeta_A/\zeta)^{1/2}\{\alpha_F(\zeta_A) - \xi_A\}e^{-i2\zeta_r Z(\zeta/\zeta_A)} \quad (38)$$

where

$$\xi = \alpha_T(\zeta)/(1 - \zeta^{*-2}),\ \xi_A = \alpha_T(\zeta_A)/(1 - \zeta_A{}^{*-2}) \quad (39)$$

and

$$Z(\zeta/\zeta_A) = \ln(\zeta/\zeta_A) + \zeta_A{}^* - \zeta^* \quad (40)$$

To facilitate the use of Eq. (38) $\alpha_F(\zeta_A)$ can be replaced by its absolute value for high spin rates and a trim history which does not change sign in the stationary region, since $\alpha_F(\zeta_A)$ is approximately a real number. This conclusion can be obtained by evaluating Eq. (14) at ζ_A using Eq. (19) from $\zeta_1 = 0$ to $\zeta_2 = \zeta_r$ and again from $\zeta_1 = \zeta_r$ to $\zeta_2 = \zeta_A$ for the integral containing the stationary point and the second integral using Eq. (17). The result is

$$\alpha_F(\zeta_A) \sim -i\zeta_A{}^{-1/2}\{(\pi/2)^{1/2}\zeta_r\alpha_T(\zeta_r)(1 - i)e^{-i2\zeta_r Z(\zeta_A{}^*)} + i[\alpha_T(\zeta_A)\zeta_A{}^{3/2}]/2(\zeta_r + \zeta_A)\} \quad (41)$$

The minimum limit of Eq. (34) is approached for large ζ_r and the exponential becomes $(i - 1)/2^{1/2}$ resulting in

$$\alpha_F(\zeta_A) \approx |\alpha_F(\zeta_A)| \quad (42)$$

for positive trim. Therefore, using Eq. (42) in Eq. (38) the absolute value of angle of attack in the poststationary region ($\zeta \geq \zeta_A$) is given by

$$|\alpha_F(\zeta)|^2 \approx \xi^2 + (\zeta_A/\zeta)[|\alpha_F(\zeta_A)| - \xi_A]^2 + 2(\zeta_A/\zeta)^{1/2}\xi[|\alpha_F(\zeta_A)| - \xi_A]\cos 2\zeta_r Z(\zeta/\zeta_A) \quad (43)$$

The instantaneous frequency of $|\alpha_F(\zeta)|$ at any altitude can be approximated by using two terms of Eq. (24) in the argument of the cosine resulting in

$$\mu_F(\zeta) = (1 - \zeta_r/\zeta)/\pi \quad (44)$$

and

$$\mu_F(t) = p_0(1 - \lambda)\mu_F(\zeta)\zeta^*/4 \quad (45)$$

If the terms other than the cosine function do not vary rapidly relative to the cosine, the maximum and minimum bounds on the angle of attack can be approximated by setting

$$\cos 2\zeta_r Z(\zeta/\zeta_A) = \pm 1 \quad (46)$$

resulting in

$$|\alpha_F(\zeta)|_{\substack{\max\\ \min}} \approx |\xi \pm (\zeta_A/\zeta)^{1/2}[|\alpha_F(\zeta_A)| - \xi_A]| \quad (47)$$

for positive trim where the sum on the right provides the maximum and the difference provides the minimum.

If all terms vary rapidly the equality in Eq. (47) is replaced by an inequality and $\alpha_F(\zeta)$ is less than the right-hand side for the positive sign in the right side and greater than the right side for the negative sign.

Results using Eqs. (38, 43, and 47) are shown in Figs. 2 and 3. The value used for $\alpha_F(\zeta_A)$ was obtained from Eq. (26), neglecting the nonstationary term according to Eq. (28), at the value ζ_A defined by Eq. (35). The differences between the present analysis and the data of Ref. 3 are primarily due to the simplifying approximations made in establishing the initial point $\alpha_F(\zeta_A)$ and ζ_A. The accuracy of the initial point can be improved as described in the discussion of the stationary region solutions.

Trajectory Dynamics Including α_E and ω_E

The angle-of-attack history of a trimmed, spinning body which is initially misaligned with the flight path and has an initial tumble rate can be obtained from Eq. (3). The high-altitude motion is given in Eq. (6) and the low-altitude angle-of-attack history is obtained by combining the low-altitude homogeneous solution with the solution providing the resonant contribution.

The low-altitude homogeneous solution,[4] in a form consistent with present notation, is

$$\alpha_H(\zeta) = (2\pi\zeta)^{-1/2}\sinh(\pi K_3/2) \times \epsilon^{-i2\zeta_r Z(-\zeta)}\{A(1 - i) + Be^{-i4\zeta}(1 + i)\} \quad (48)$$

where the Bessel functions were approximated using Eq. (13) and the constants A and B are given by Eqs. (8) and (9).

The combined angle-of-attack history can be obtained by adding Eq. (48) to either Eqs. (18, 26 or 38) for the prestationary, stationary or poststationary region, respectively. Simplified relations are required, however, in order to obtain an insight as to the significant behavior of the combined angle-of-attack history throughout the trajectory. The discussion of trajectory dynamics is therefore subdivided according to bounding angle-of-attack histories using the general homogeneous equation, and detailed angle-of-attack histories of a trimmed body entering with initial misalignment and zero tumble rate.

Bounding Values of $\alpha(\zeta)$

The bounds on the angle-of-attack history including both initial misalignment and tumble rate can be relatively easily defined.[5] The absolute value of angle of attack in any region is bounded by

$$|\alpha(\zeta)|_{\max} \leq |\alpha_F(\zeta)| + |\alpha_H(\zeta)| \quad (49)$$

and

$$|\alpha(\zeta)|_{\min} \geq ||\alpha_F(\zeta)| - |\alpha_H(\zeta)|| \quad (50)$$

The absolute value of the homogeneous contribution is obtained from Eq. (48) as

$$|\alpha_H(\zeta)|^2 = (K_3/2)\zeta\{|\omega_E/p_0(1 + \lambda)|^2 + |\alpha_E + i\omega_E/p_0(1 + \lambda)|^2 + 2|\omega_E/p_0(1 + \lambda)||\alpha_E + i\omega_E/p_0(1 + \lambda)|\sin(4\zeta + \nu)\} \quad (51)$$

where

$$\nu = \tan^{-1}[(A_{\mathcal{I}}B_{\mathcal{R}} - A_{\mathcal{R}}B_{\mathcal{I}})/(A_{\mathcal{R}}B_{\mathcal{R}} + A_{\mathcal{I}}B_{\mathcal{I}})] \quad (52)$$

The instantaneous frequency is a constant per unit ζ, as given by

$$\mu_H(\zeta) = 2/\pi \quad (53)$$

and by using two terms of Eq. (24)

$$\mu_H(t) = \zeta^*p_0(1 - \lambda)/2\pi \quad (54)$$

Bounding values for the maximums and minimums of the homogeneous contribution, Eq. (51), can be approximated by setting

$$\sin(4\zeta + \nu) = \pm 1 \quad (55)$$

respectively. The maximums and minimums are

$$|\boldsymbol{\alpha}_H(\zeta)|_{\substack{\max\\ \min}} \approx (K_3/2\zeta)^{1/2}|\{|\omega_E/p_0(1 + \lambda)| \pm |\alpha_E + i\omega_E/p_0(1 + \lambda)|\} \quad (56)$$

where the maximum is given by the sum and the minimum by the difference. These approximations Eq. (56) will differ from the actual values due to the multiplier $\zeta^{-1/2}$.

The bounding resonant contribution is required for only the stationary and poststationary regions since the prestationary solution, Eq. (18) is already in convenient form. The comparison, Fig. 3, suggests that a reasonable estimate of the maximum value of $|\alpha_F(\zeta)|$ in the region $\zeta_r \leq \zeta \leq \zeta_A$, can be obtained by using two terms of Eq. (24) in Eq. (22) and neglecting the integral with no stationary point as shown in Eqs. (29) and (30). The absolute magnitude of $\alpha_F(\zeta)$ in this region, including the first maximum, can therefore be bounded by

$$|\alpha_F(\zeta)| \leq \zeta^{-1/2}\left\{\left|\left(\frac{\pi}{4}\right)^{1/2}\zeta_r\alpha_T(\zeta_r)\right| + \left|\int_{\zeta_r}^{\zeta}\alpha_T(\zeta)\zeta^{1/2}d\zeta\right|\right\} \quad (57)$$

and for constant trim

$$|\alpha_F(\zeta)/\alpha_{T0}| \leq \zeta^{-1/2}\{(\pi/4)^{1/2}\zeta_r + \tfrac{2}{3}(\zeta^{3/2} - \zeta_r^{3/2})\} \quad (58)$$

Equation (58) is compared with the data of Ref. 3 in Fig. 8 showing that the bound is valid except for low ζ_r which is expected.

The bounding values of $\alpha_F(\zeta)$ in the poststationary region are given by Eq. (47).

Zero Tumble Rate

The specific case $\omega_E = 0$ defines a spiral type of motion and is also of general interest, allowing insight concerning the significant parameters which effect trajectory dynamics.

For the case $\omega_E = 0$, Eq. (48) becomes

$$\alpha_H(\zeta) = (K_3/2\zeta)^{1/2}|\alpha_E|\{\sin g(\zeta) - i\cos g(\zeta)\} \quad (59)$$

and the absolute value is

$$|\alpha_H(\zeta)| = |\alpha_E|(K_3/2\zeta)^{1/2} \quad (60)$$

where

$$g(\zeta) = 2\zeta_r Z(+\zeta) + \theta_B \quad (61)$$

and

$$\theta_B = \tan^{-1}[(B_{\Re} - B_{\Im})/(B_{\Re} + B_{\Im})] \quad (62)$$

The trimmed body dynamics are obtained from

$$\alpha(\zeta) = \alpha_{F_{\Re}}(\zeta) + |\alpha_H(\zeta)|\sin g(\zeta) + i[\alpha_{F_{\Re}}(\zeta) - |\varphi_H(\zeta)|\cos g(\zeta)] \quad (63)$$

and the absolute value from

$$|\alpha(\zeta)|^2 = |\alpha_F(\zeta)|^2 + |\alpha_H(\zeta)| + 2|\alpha_F(\zeta)||\alpha_H(\zeta)|\sin[g(\zeta) + \theta_F] \quad (64)$$

$$\theta_F = \tan^{-1}[-\alpha_{F_{\Im}}(\zeta)/\alpha_{F_{\Re}}(\zeta)] \quad (65)$$

The bounds on $\alpha(\zeta)$ are obtained by using Eq. (60) in Eqs. (49) and (50) and the comments relative to the validity of the use of Eq. (55) to define the maximums and minimums apply here to the sine function in Eq. (64).

The value of θ_F is zero in the prestationary region, as shown by Eq. (18), at $\zeta = \zeta_A$ from Eq. (42), and approaches zero in the poststationary region, since Eq. (38) shows that $\alpha_F(\zeta)$ approaches $\alpha_T(\zeta)$ for large ζ. Therefore, for no tumble rate,

$$|\alpha(\zeta)|^2 = \xi^2 + (K_3/2\zeta)|\alpha_E|^2 + 2(K_3/2\zeta)^{1/2}|\alpha_E|\xi^2 \sin g(\zeta) \quad (66)$$

in the prestationary region indicating that the homogeneous oscillatory contribution is simply superimposed on the amplified trim contribution. If the terms other than the sine functions do not vary rapidly, the equalities shown in Eqs. (49) and (50) are valid.

The frequency can be approximated by using two terms of Eq. (24) in the argument of the sine resulting in

$$\mu(\zeta) = (\zeta^{*-1} + 1)/\pi \quad (67)$$

or

$$\mu(t) = p_0(1 - \lambda)\mu(\zeta)\zeta^*/4 \quad (68)$$

The behavior of $|\alpha(\zeta)|$ near the altitude of maximum trim amplification is obtained by substituting Eq. (43) into Eq. (64). The change in the cyclic contribution of Eq. (43) is small near ζ_A. By using two terms of the binomial series for $|\alpha_F(\zeta)|$ and two terms of Eq. (24) in the argument of the trigonometric functions, $|\alpha(\zeta)|$ history near ζ_A for large ζ_r and positive trim becomes

$$|\alpha(\zeta)|^2 \approx |\alpha_F(\zeta)|^2 + |\alpha_H(\zeta)|^2 + 2|\alpha_H(\zeta)||\alpha_F(\zeta)|_{\max}\sin(4\zeta + \theta_1) \quad (69)$$

where $\theta_1 = 2\zeta_r(\ln\zeta_A - 1) + \theta_B$ and $|\alpha_F(\zeta)|_{\max}$ is given in Eq. (47).

Equation (69) shows that, near the altitude represented by ζ_A, $|\alpha(\zeta)|$ has a trim amplification contribution and a convergence contribution. The cyclic nature is made up of a low-frequency contribution given by Eqs. (44) or (45) and a high-frequency contribution, Eqs. (53) or (54). Since the low frequency is near zero the motion of $|\alpha(\zeta)|$ may be visualized as convergent or divergent depending on the relative magnitude of the trim and convergent contributions and oscillating with a high frequency. In addition, a line passing through the points of maximum $|\alpha(\zeta)|$, would also exhibit a sinusoidal oscillation with a low frequency as defined by Eqs. (44) or (45) and shown in Ref. 5. Finally, the large difference in frequencies offer the conclusion that the altitude of the first maximum angle of attack after ζ_r for positive trim can be perturbed by at most a half wavelength of the high-frequency contribution, or between $\zeta_M - \pi/4$, and $\zeta_M + \pi/4$, because of initial misalignment, where ζ_M is given by Eq. (33) for the perfectly aligned body.

Insight into poststationary region behavior for $\zeta \gg \zeta_A$ may be obtained by again using Eqs. (24, 43 and 64). The oscillation frequencies are obtained by assuming that $\zeta_A/\zeta \ll 1$ and the contributions to $|\alpha_F(\zeta)|_{\max}$ and $|\alpha_H(\zeta)|$ change insignificantly in one cycle of oscillation. Following the approach described for Eq. (69) the resulting high-frequency oscillation is identical with the value given by Eqs. (53) or (54); however, the low-frequency oscillation is

$$\mu(\zeta \gg \zeta_A) \approx 1/\pi \quad (70)$$

which is the same result as obtained from Eq. (44) by taking the limit as $\zeta \to \infty$.

Conclusion

The results of the analysis and the numerical comparisons indicate that by applying the method of stationary phase, an insight into the dynamic behavior of a trimmed spinning body can be obtained.

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